NASA TECHNICAL NOTE

NASA TN D-4518

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ANALYTICAL CALCULATION OF PARTIAL DERIVATIVES RELATING LUNAR AND PLANETARY MIDCOURSE CORRECTION REQUIREMENTS TO GUIDANCE SYSTEM INJECTION ERRORS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1968

ERRATA

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Page 6: The last of equations (6) should read

$$t - t_p = \sqrt{\frac{p^3}{\mu}}\left(\frac{1}{1 - e^2}\right)\left[\sqrt{\frac{1}{e^2 - 1}}\ \log\left(\frac{\sqrt{e + 1} + \sqrt{e - 1}\ \tan\frac{\eta}{2}}{\sqrt{e + 1} - \sqrt{e - 1}\ \tan\frac{\eta}{2}}\right) - \frac{e \sin \eta}{1 + e \cos \eta}\right]$$

Page 10: The time from perigee equation should read

$$t - t_p = \sqrt{\frac{p^3}{\mu}}\left[\frac{1}{(1 - e)^2}\right]\left[\frac{2}{\sqrt{1 - e^2}}\ \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\ \tan\frac{\eta}{2}\right) - \frac{e \sin \eta}{1 + e \cos \eta}\right]$$

Page 31: The second last of equations (C1) should read

$$t_I - t_p = \sqrt{\frac{p^3}{\mu}}\left(\frac{1}{1 - e^2}\right)\left[\sqrt{\frac{1}{e^2 - 1}}\ \log\left(\frac{\sqrt{e + 1} + \sqrt{e - 1}\ \tan\frac{\eta_I}{2}}{\sqrt{e + 1} - \sqrt{e - 1}\ \tan\frac{\eta_I}{2}}\right) - \frac{e \sin \eta_I}{1 + e \cos \eta_I}\right]$$

Page 33: Equation (C5) should read

$$t_I - t_p = \sqrt{\frac{p^3}{\mu}}\left(\frac{1}{1 - e^2}\right)\left[\frac{2}{\sqrt{1 - e^2}}\ \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\ \tan\frac{\eta_I}{2}\right) - \frac{e \sin \eta_I}{1 + e \cos \eta_I}\right]$$

NASA-Langley, 1968

Issued 7-30-68

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SUMMARY

This report describes a simplified analytical method for calculating partial derivatives relating midcourse correction velocity requirements to injection errors resulting from guidance system errors for lunar and planetary missions.

The analytical method uses two-body equations of motion to describe the reference transfer orbit. These equations are linearized to obtain partial derivatives relating changes in terminal state conditions to changes at injection and midcourse. The partial derivatives are then used to calculate the required correction velocity.

The analytical equations are easily programmed on a digital computer and require very little computer time. The only input variables required are the position and velocity components at injection from a reference trajectory.

The analytical partial derivatives are compared to exact partial derivatives obtained from a detailed N Body computer simulation. A comparison of results is presented for two lunar and five planetary missions. The error in the approximate results is found to be less than 14 percent for all cases simulated.

INTRODUCTION

The predicted guidance system performance for a space mission is usually measured in terms of a single number, called the figure of merit (FOM). For lunar and planetary probes, the FOM is the expected value of the midcourse velocity correction, which is used to correct miss plus time of flight or miss only errors at the target.

The FOM may be calculated as the product of two matrices: the injection covariance matrix (ICM) and the FOM covariance matrix. The ICM gives a complete statistical description of the injection errors and may be obtained by integration of the adjoint of the equations of motion from launch to termination of powered flight for each guidance system error source. The FOM covariance matrix consists of a set of partial derivatives relating FOM to the injection errors. The exact calculation of these partial derivatives is a tedious one requiring an N Body computer program and much computer time. The purpose of this report is to derive an approximate analytical method which allows these partial derivatives and, hence the FOM covariance matrix, to be easily calculated.

In preliminary design studies, the mission planner may wish to investigate the FOM requirements for various combinations of trajectories, launch vehicles and/or guidance hardware error models. In such cases, the detailed mission requirements and launch opportunities usually have not been established; and targeted reference trajectories are not available. Also, the mission planner may not have a detailed N Body code at his disposal, and therefore could not calculate exact (N Body) partial derivatives. It is for this type of problem that the authors decided to investigate possible analytical methods for generating the partial derivatives required to compute the FOM. The analytical methods would be easy to implement on a computer and require little computer time.

As a result of this study, equations were developed for computing analytical partial derivatives of FOM with respect to injection state variables for planetary and lunar probes. The analytical equations for computing these derivatives are the same for the two mission types, but the approach used is somewhat different. In both cases, changes in arrival conditions at the target are related to injection errors and to midcourse corrections by using a method described by Danby in references 1 and 2. Basically, Danby's method relates errors between any two points on a two-body orbit by using a first-order analysis.

A digital computer program was written to test the analytical equations and to compare the analytical results to the detailed N Body results. A reference powered flight trajectory is required as input to the program. In addition, a typical guidance hardware error model and error values were assumed. These errors provide the forcing function for the linearized equations of motion, which are integrated in the program to obtain the injection covariance matrix. The reference transfer orbit is obtained by using the injection conditions from the reference powered flight trajectory in the two-body analytical equations.

Analytical and N Body results are presented for lunar and planetary missions, including Venus, Mars, Mercury, and Jupiter. The results presented include the analytical and N Body FOM for each error source, as well as the total FOM. The analytical and N Body FOM covariance matrices are also presented. A sample problem is presented wherein the trajectory resulting in the largest FOM for a Mars mission is determined by using the analytical technique. This problem serves to illustrate the use of the analytical technique.

ANALYSIS

General Equations for FOM

For planetary and lunar probe missions, the function of the launch vehicle guidance system is to steer the vehicle during its powered phases and to command cutoff, such that the resulting free flight trajectory will arrive at the desired target, usually at some prespecified time. The guidance system performance for these missions can be specified in terms of a single number, called the figure of merit (FOM). The FOM is the expected value of the midcourse correction velocity, required to correct miss plus time of flight errors at the target body or to correct miss only, if time of flight errors are not important.

For a definite set of injection errors, the required midcourse correction can be determined by simulating the dispersed trajectory to the target, and iterating on the midcourse correction to zero the error in arrival conditions. Since injection position and velocity errors due to guidance are usually small, a first-order analysis has generally been used to propagate state vector deviations from injection and midcourse to the target. The use of first-order equations also allows the effects of various guidance system error sources to be statistically combined to obtain the FOM.

If a first-order analysis is used, the midcourse correction for a set of injection errors can be expressed as

$$\Delta\bar{v}_c = G\delta\bar{S} \tag{1}$$

where G is a 3×6 matrix of partial derivatives

$$G_{ik} = \frac{\partial\left(\Delta v_c\right)_i}{\partial S_k} \qquad \begin{array}{l} i = 1,\ 2,\ 3 \\ k = 1,\ .\ .\ .,\ 6 \end{array}$$

and

$$\delta\bar{S} = \begin{pmatrix} \delta\bar{r}_I \\ \delta\bar{v}_I \end{pmatrix}$$

Injection is defined here as the time of termination of powered flight.

The correction velocity squared can be expressed as:

$$\left(\Delta v_c\right)^2 = \Delta\bar{v}_c^T\ \Delta\bar{v}_c = \left(G\delta\bar{S}\right)^T\left(G\delta\bar{S}\right) \tag{2}$$

where T denotes the transpose of a matrix. Equation (2) may be rewritten as:

$$\left(\Delta v_c\right)^2 = \delta\bar{S}^T G^T G \delta\bar{S} \tag{3}$$

The 6×6 matrix $G^T G$ is defined as the FOM covariance matrix

$$\Lambda = G^T G$$

By replacing $G^T G$ with Λ, equation (3) may be rewritten in summation form as:

$$\left(\Delta v_c\right)^2 = \sum_{i=1}^{6} \sum_{j=1}^{6} \Lambda_{ij} \delta S_i \delta S_j \tag{4}$$

The expected value operator E can be applied to equation (4) to give:

$$(\mathrm{FOM})^2 = E\left[\left(\Delta v_c\right)^2\right] = \sum_{i=1}^{6} \sum_{j=1}^{6} \Lambda_{ij} \sigma_{ij} \tag{5}$$

where

$$\sigma_{ij} = E(\delta S_i \delta S_j)$$

The same Λ matrix applies for either a definite or a statistical set of injection errors.

Calculation of FOM Covariance Matrix - Planetary Mission

For a planetary mission, the FOM is the midcourse velocity required to correct miss plus time of flight errors at the target planet. The correction is usually performed several days after injection, after enough tracking data has been obtained to define the transfer orbit.

For the analytical calculations, a series of two-body transfer conics is assumed, with the transition from the earth's field to heliocentric space occurring at the earth's sphere of influence. The midcourse correction is assumed to take place at the earth's sphere of influence. This gives a correction time of from 1 to 4 days (depending on the injection energy) which is consistent with the real case. The following assumptions are also made:

(1) The error in time at earth escape is negligible compared to the total trip time.

(2) The position error at earth escape is negligible compared to heliocentric dimensions.

With these assumptions, the effect of injection position and velocity errors is simply to change the magnitude and direction of the hyperbolic velocity vector. Therefore, the midcourse correction velocity is equal to the error in the hyperbolic velocity vector.

The first step in the analysis is to compute the reference trajectory parameters at injection. All reference orbit parameters in this report are calculated from the two-body equations tabulated in reference 3. Some of the orbit parameters defined in equation (6) are illustrated in figure 1.

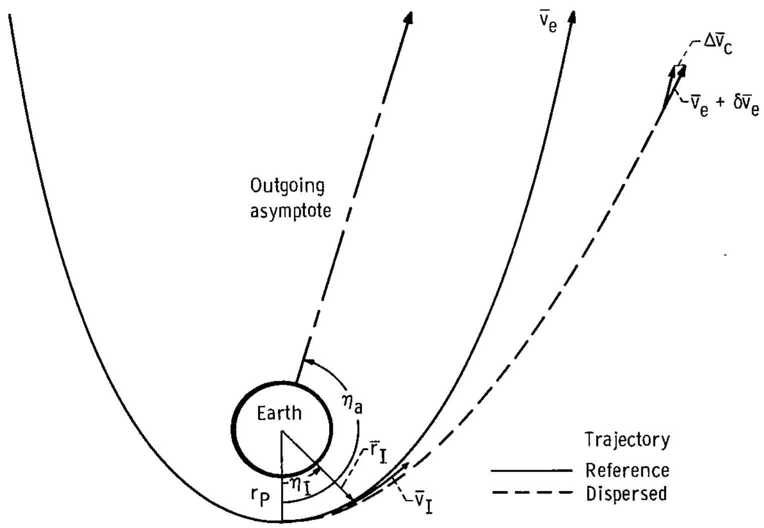

Figure 1. - Trajectory geometry for Earth-centered phase of planetary mission.

Angular momentum:

$$\bar{h} = \bar{r} \times \bar{v}$$

Similatus rectum:

$$p = \frac{\bar{h} \cdot \bar{h}}{\mu}$$

Energy:

$$E = \frac{\bar{v} \cdot \bar{v}}{2} - \frac{\mu}{r}$$

Eccentricity:

$$e = \sqrt{1 + \frac{2Ep}{\mu}}$$

Perigee radius:

$$r_p = \frac{p}{1 + e}$$

Injection true anomaly:

$$\eta_I = \sin^{-1}\left(\sqrt{\frac{p}{\mu}}\ \frac{\bar{r} \cdot \bar{v}}{er}\right)$$

Asymptote true anomaly:

$$\eta_a = \cos^{-1}\left(-\frac{1}{e}\right)$$

Time from perigee:

$$t - t_p = \sqrt{\frac{p^3}{\mu}}\left(\frac{1}{1 - e^2}\right)\sqrt{\frac{1}{e^2 - 1}}\left[\log\left(\frac{\sqrt{e + 1} + \sqrt{e - 1}\tan\frac{\eta}{2}}{\sqrt{e + 1} - \sqrt{e - 1}\tan\frac{\eta}{2}}\right) - \frac{e \sin \eta}{1 + e \cos \eta}\right] \tag{6}$$

All symbols are defined in appendix A.

The error in the hyperbolic velocity vector is calculated by Danby's method as described in references 1 and 2. Danby's method is briefly described in this report in appendix B.

For the planetary case, interest lies in calculating the $\delta\bar{v}$ at earth escape.

$$\Delta\bar{v}_c = \delta\bar{v}_e = C_{eI}\delta\bar{r}_I + D_{eI}\delta\bar{v}_I \tag{7}$$

where $C_{eI} = C(t_e, t_I)$ and $D_{eI} = D(t_e, t_I)$ are calculated from equation (B2) in appendix B.

The form of equation (1) has been satisfied by equation (7) by combining C and D into the correction sensitivity matrix G:

$$G = (C|D)$$

The Λ matrix is then given by

$$\Lambda = G^T G$$

By examining the equations in appendix B, it will be noted that the L and M matrices go to infinity as time approaches infinity. However, the P and Q matrices remain finite. Physically, this means that the position errors increase without bound for any injection errors. However, in keeping with the assumptions made earlier, the position errors are assumed negligible with respect to heliocentric dimensions and assumed to be zero for this case. This will be a good assumption if the hyperbolic velocity is large so that earth escape occurs quickly and position errors do not have time to build up appreciably.

It is necessary to compute the unit radial coordinate system at perigee since Danby's equations are referenced to perigee. This is done as follows:

$$\hat{r}_p = \frac{\bar{v} \times \bar{h}}{\mu e} - \frac{\bar{r}}{er}$$

$$\hat{h}_p = \frac{\bar{h}}{h}$$

$$\hat{\theta}_p = \hat{h}_p \times \hat{r}_p$$

where $\bar{r}$, $\bar{v}$, and $\bar{h}$ are determined at injection. All state vector deviations must be expressed in this coordinate system.

The complete set of equations required to calculate the Λ matrix for the planetary case is presented in appendix C.

Calculation of FOM Covariance Matrix - Lunar Mission

The FOM for the lunar case is the expected value of the midcourse velocity required to correct either miss plus time of flight to the moon or miss only. This correction is usually made about 15 to 20 hours after injection, but can be computed for any desired correction time by using either the exact or analytical methods.

The Δv_c is calculated by using Danby's matrices - the same technique that is used in the planetary case and described in appendix B. The miss at the moon is calculated at the incoming asymptote at the lunar sphere of influence, relative to the reference geocentric conic trajectory. In this case, the following definitions are made:

$$t_c = \text{time of midcourse correction}$$

$$t_A = \text{time of lunar arrival}$$

The uncorrected miss at the moon is given by:

$$\delta\bar{r}_m = A_{mI}\delta\bar{r}_I + B_{mI}\delta\bar{v}_I + (\bar{v}_A - \bar{v}_m)\delta t_A$$

where $A_{mI} = A(t_m, t_I)$ and $B_{mI} = B(t_m, t_I)$ are calculated from equation (B2) in appendix B.

$$\bar{v}_A = \text{arrival velocity at the moon}$$

$$\bar{v}_m = \text{moon's orbital velocity about the earth}$$

$$\delta t_A = \text{error in arrival time}$$

The change in arrival position due to midcourse correction is:

$$\delta\bar{r}_m = B_{mc}\delta\bar{v}_c$$

To correct miss plus time, the time error δt_A must be zero. Therefore,

$$\Delta\bar{v}_{c,MT} = -B_{mc}^{-1}\left(A_{mI}\delta\bar{r}_I + B_{mI}\delta\bar{v}_I\right) = G\delta\bar{S} \equiv \bar{\bar{U}} \tag{8}$$

where

$$G = -\left(B_{mc}^{-1}A_{mI} \middle| B_{mc}^{-1}B_{mI}\right)$$

Therefore,

$$\Lambda_{MT} = G^T G$$

To correct miss only, the arrival time t_A may be optimized to give minimum Δv_c.

$$\Delta\overline{v}_{c,M} = \overline{U} - B_{mc}^{-1}(\overline{v}_A - \overline{v}_m)\delta t_A = \overline{U} + \overline{V}\delta t_A \tag{9}$$

where

$$\overline{V} = -B_{mc}^{-1}(\overline{v}_A - \overline{v}_m) \tag{10}$$

The following is obtained by squaring equation (9):

$$\left(\Delta v_{c,M}\right)^2 = \overline{U}^T\overline{U} + 2\overline{U}^T\overline{V}\delta t_A + \overline{V}^T\overline{V}\delta t_A^2$$

For optimum t_A,

$$\frac{\partial\left(\Delta v_{c,M}\right)^2}{\partial t_A} = 2\left(\overline{U}^T\overline{V} + \overline{V}^T\overline{V}\delta t_A\right) = 0$$

Therefore,

$$\delta t_A = -\frac{\overline{U}^T\overline{V}}{\overline{V}^T\overline{V}}$$

and

$$\left(\Delta v_{c,M}\right)^2 = \overline{U}^T\overline{U} - \frac{\left(\overline{U}^T\overline{V}\right)^2}{\overline{V}^T\overline{V}}$$

The Λ matrix can be constructed for the miss only case by using equation (8):

$$\left(\Delta v_{c,M}\right)^2 = \delta\bar{S}^T\left(G^TG - \frac{G^T\bar{V}\bar{V}^TG}{\bar{V}^T\bar{V}}\right)\delta\bar{S}$$

where G refers to the miss plus time matrix; therefore, from equation (3),

$$\Lambda_M = \Lambda_{MT} - \frac{G^T\bar{V}\bar{V}^TG}{\bar{V}^T\bar{V}}$$

for the miss only case.

As in the planetary case, the elements of the A and B matrices are calculated from the equations in appendix B. The reference trajectory parameters are the same as for the planetary case (eq. (6)), except that true anomaly at lunar arrival is used in place of true anomaly at the outgoing asymptote.

True anomaly at lunar arrival:

$$\eta_A = \cos^{-1}\left(\frac{\frac{p}{r_m} - 1}{e}\right)$$

The time from perigee equation is:

$$t - t_p = \sqrt{\frac{p^3}{\mu}}\left[\frac{1}{(1 - e)^2}\right]\frac{2}{\sqrt{1 - e^2}}\left[\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\eta}{2}\right) - \frac{e \sin \eta}{1 + e \cos \eta}\right]$$

Some of the trajectory parameters are illustrated in figure 2. For the lunar mission, $\bar{v}_A$ and $\bar{v}_m$ are calculated as follows:

Unit vector to the moon:

$$\hat{r}_A = \cos \eta_A \hat{r}_p + \sin \eta_A \hat{\theta}_p$$

Velocity at lunar arrival:

$$\bar{v}_A = \frac{e}{p}\bar{h} \times \hat{r}_p + \frac{\bar{h}}{p} \times \hat{r}_A$$

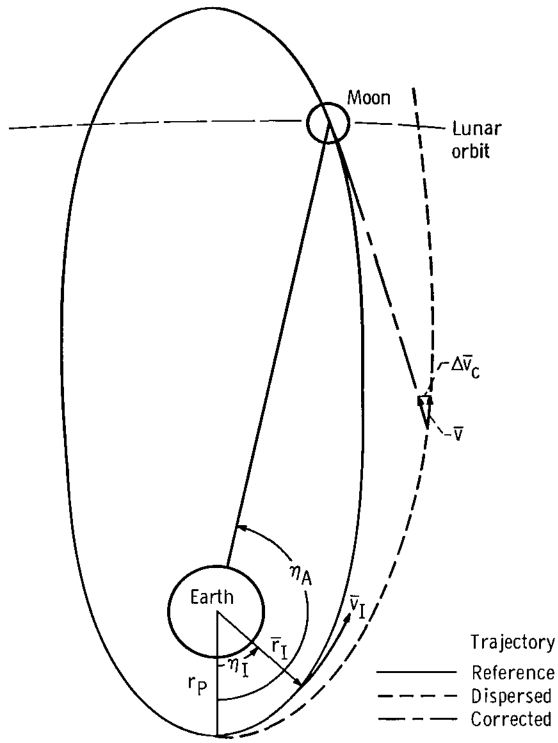

Figure 2. - Lunar trajectory geometry.

To calculate the lunar velocity, the following are assumed:

(1) Lunar orbit is circular about the earth.

(2) ψ is the inclination of the moon's orbit to the earth's equator.

As shown in appendix D, these assumptions result in

$$\bar{v}_m = \sqrt{\frac{\mu}{r_m}}\,\frac{1}{\cos\varphi}\left\{-\frac{\cos\psi}{\cos\varphi}(\hat{r}_m \times \hat{z}) \pm \sqrt{1 - \frac{\cos^2\psi}{\cos^2\varphi}}\left[\hat{r}_m \times (\hat{r}_m \times \hat{z})\right]\right\}$$

where φ is the latitude of the moon at arrival:

$$\sin\varphi = \hat{r}_m \cdot \hat{z}$$

The choice of sign in $\bar{v}_m$ depends on whether the moon is ascending or descending:

+ sign indicates descending moon

- sign indicates ascending moon

The complete set of equations required to calculate the Λ matrix is presented in appendix C.

Calculation of Injection Errors

In order to test the analytical equations for FOM, it was necessary to generate an injection covariance matrix. This was done by choosing a typical guidance hardware error model and integrating the linearized equations of motion.

Guidance hardware error model. - A typical platform inertial guidance system hardware error model was chosen and is described in table I. Among the errors considered were accelerometer bias and scale factor, gyro bias and g-sensitive drift and platform misalinement. The errors, when applied to the nominal thrust acceleration profile, provide the forcing function $\delta\bar{a}$ for the linearized equations of motion. The platform is alined along the u, v, w inertial coordinate system which is established at liftoff. The u axis points in the pad azimuth direction. The w axis lies along the local plumbline and v completes the right-handed orthogonal set.

Linearized equations of motion. - The equations of motion for the reference trajectory in vector form are:

$$\ddot{\bar{r}} = -\frac{\mu\bar{r}}{r^3} + \bar{a} \tag{11}$$

where $\ddot{\bar{r}}$ is the total acceleration, $-\mu\bar{r}/r^3$ is the gravity acceleration, and $\bar{a}$ is the thrust acceleration. If the error in sensed acceleration $\delta\bar{a}$ is small, then the error in total acceleration (i.e., difference between measured and actual acceleration) may be obtained by linearizing equation (11). The result is:

$$\delta\ddot{\bar{r}} = -\frac{\mu}{r^3}\,\delta\bar{r} + \frac{3\mu}{r^5}\,(\bar{r}\cdot\delta\bar{r})\bar{r} + \delta\bar{a} \tag{12}$$

The difference between the measured and actual state vectors, $\delta\bar{S}$, is obtained by integrating equation (12).

Injection errors. - The injection state vector error is defined by

$$\Delta\bar{S} = \bar{S}_{act} - \bar{S}_{ref} \equiv -\delta\bar{S} + (\bar{S}_{mes} - \bar{S}_{ref}) \tag{13}$$

where the subscripts (act, ref, mes) refer to the (actual, reference, measured by the

TABLE I. - DESCRIPTION OF HARDWARE ERROR MODEL

Error source	Error symbol	Description	Error value	Error source	Error symbol	Description	Error value
Initial platform misalinement, mrad	e_1	u Axis	0.054	Absolute accelerometer scale factor, percent	e_{25}	u Accelerometer	0
	e_2	v Axis	.054		e_{26}	v Accelerometer	0
	e_3	w Axis	.090		e_{27}	w Accelerometer	0
Accelerometer input axis misalinement, mrad	e_4	v Relative to u	0.050	Accelerometer scale factor (quadratic term), mg/g^2	e_{28}	u Accelerometer	0.0094
	e_5	w Relative to u	.050		e_{29}	v Accelerometer	.0094
	e_6	w Relative to v	.055		e_{30}	w Accelerometer	.0094
Accelerometer preflight bias error, mg	e_7	u Accelerometer	0.042	Accelerometer scale factor (cubic term), mg/g^3	e_{31}	u Accelerometer	0
	e_8	v Accelerometer	.042		e_{32}	v Accelerometer	0
	e_9	w Accelerometer	.042		e_{33}	w Accelerometer	0
Accelerometer scale factor (linear), percent	e_{10}	u Accelerometer	0.051	Accelerometer cross-coupling (input-pendulous axes), mg/g^2	e_{34}	u Accelerometer	0.013
	e_{11}	v Accelerometer	.051		e_{35}	v Accelerometer	.013
	e_{12}	w Accelerometer	.051		e_{36}	w Accelerometer	.013
Gyro fixed torque drift, deg/hr	e_{13}	u Gyro	0.084	Accelerometer cross-coupling (input-output axes), mg/g^2	e_{37}	u Accelerometer	0.012
	e_{14}	v Gyro	.093		e_{38}	v Accelerometer	.012
	e_{15}	w Gyro	.094		e_{39}	w Accelerometer	.012
Spin axis mass unbalance drifts, deg/(hr)(g)	e_{16}	u Gyro	0.173	Accelerometer inflight bias error, mg	e_{40}	u Accelerometer	0.024
	e_{17}	v Gyro	.190		e_{41}	v Accelerometer	.026
	e_{18}	w Gyro	.177		e_{42}	w Accelerometer	.029
Input axis mass unbalance drifts, deg/(hr)(g)	e_{19}	u Gyro	0.106	Accelerometer bias error output axis loading (quadratic), mg/g^2	e_{43}	u Accelerometer	0.009
	e_{20}	v Gyro	.101		e_{44}	v Accelerometer	.009
	e_{21}	w Gyro	.114		e_{45}	w Accelerometer	.009
Anisoelastic drifts, $deg/(hr)(g^2)$	e_{22}	u Gyro	0.009	Accelerometer cross-coupling (pendulous-output axes), mg/g^2	e_{46}	u Accelerometer	0.008
	e_{23}	v Gyro	.009		e_{47}	v Accelerometer	.008
	e_{24}	w Gyro	.009		e_{48}	w Accelerometer	.008

guidance system) trajectories. If a guidance cutoff is used and perfect guidance software is assumed, then the midcourse correction requirement results solely from hardware errors, and

$$\Delta\bar{v}_c = G\,\Delta\bar{S} = -G\delta\bar{S}$$

Therefore, the midcourse correction may be obtained by applying G to the $\delta\bar{S}$ obtained by integration of equation (12). A guidance cutoff will normally be used for the lunar and planetary missions in order to obtain the desired injection energy.

Calculation of Injection Covariance Matrix

The 6×6 matrix σ is called the injection covariance matrix (ICM). The injection errors result from errors in the guidance system accelerometers and gyros. Since the expected hardware errors are of a statistical nature, the elements of the ICM are defined as the expected values of the products of the state variable deviations; that is:

$$\sigma_{ij} = E(\delta S_i \delta S_j) \qquad \begin{array}{l} i = 1, \ldots, 6 \\ j = 1, \ldots, 6 \end{array}$$

The state variables S_i are the reference trajectory position and velocity components at injection and the δS_i are the corresponding deviations in these state variables.

For each guidance system error source, the injection state deviation can be expressed as:

$$\delta S_i = P_{ik} e_k \qquad i = 1, \ldots, 6$$

where e_k is the magnitude of the k^{th} error and P_{ik} is the matrix of partial derivatives $\partial S_i / \partial e_k$. The elements of the injection covariance matrix (for N guidance system error sources) are given by:

$$\sigma_{ij} = E(\delta S_i \delta S_j) = \sum_{k=1}^{N} \sum_{l=1}^{N} P_{ik} P_{jl} E(e_k e_l)$$

If the various error sources are assumed to be independent, then the elements of the injection covariance matrix become:

$$\sigma_{ij} = \sum_{k=1}^{N} (P_{ik}P_{jk})E\left(e_k^2\right) \tag{14}$$

The total $(FOM)^2$ is obtained by combining equations (5) and (14).

$$(FOM)^2 = \sum_{k=1}^{N}\left(\sum_{i=1}^{6}\sum_{j=1}^{6} \Lambda_{ij}P_{ik}P_{jk}\right)E\left(e_k^2\right) \tag{15}$$

For any given error model, the quantities in parentheses in equations (14) and (15) represent the contribution of each error source to the injection covariance matrix and to $(FOM)^2$, respectively. Once these coefficients are calculated, the ICM and $(FOM)^2$ can be obtained for any error budget by performing the indicated summations in equations (14) and (15). The contribution of the individual error sources can also be easily calculated.

In the computer program, the linearized equations of motion are integrated to obtain $\delta\bar{S}$ for each error source. A FOM is then computed for each error source. The total FOM is obtained by calculating the RSS of the individual FOM's.

RESULTS AND DISCUSSION

In order to determine the accuracy of the approximate analytical equations presented, both analytical and exact (N Body) error analysis results have been obtained and are presented here for comparison. For the N Body results, targeted powered flight and free flight trajectories were obtained by using the procedures discussed earlier. The lunar trajectories were targeted for a vertical impact with a prespecified time of flight. The planetary cases were also targeted to a prespecified flight time, but a close planetary approach was used as the targeting criteria. After the targeted reference trajectory was obtained, the linearized equations of motion were integrated from powered flight injection to the target body and from the midcourse correction point to the target. The partial derivatives obtained in this manner were then combined to give the exact FOM covariance matrix, as discussed earlier.

The analytical FOM covariance matrix is obtained by using the injection conditions from the targeted powered flight reference trajectory in the analytical equations presented earlier. It is important to note that injection conditions could have been obtained from many sources, such as references 4 and 5. As stated earlier, it is not necessary to have targeted reference trajectories in order to obtain analytical error analysis results. The targeted injection conditions are used here to insure an exact comparison between the analytical and N Body results.

The position, velocity, and acceleration profiles from the reference trajectory are required as input to the error analysis program. Several different configurations of the Atlas-Centaur launch vehicle were used to simulate the powered flight reference trajectories, as shown in table II. These configurations do not necessarily represent existing or planned launch vehicles or stages; however, the configurations used are realistic ones, and the results obtained can be considered representative.

TABLE II. - GENERAL DESCRIPTION OF TRAJECTORIES SIMULATED

Case	Mission	Ascent mode	Launch date	Launch azimuth, deg	Energy, km^2/sec^2	True anomaly, deg	Parking orbit coast time, sec	Launch vehicle
1	Mars	Direct	3/22/69	111	14.45	-7.54	----	Atlas-Centaur - upper stage[a]
2	Mars	Direct	12/13/66	120	15.07	16.67	----	Atlas-Centaur - upper stage[a]
3	Jupiter	Direct	2/26/72	95	114	-2.41	----	Atlas-Centaur - Burner II
4	Venus	Two-burn	8/13/70	90	15.05	-----	445	Atlas-Centaur
5	Mercury	Two-burn	11/12/68	90	45.36	-----	1232	Atlas-Centaur
6	Lunar	Direct	6/3/66	93	-1.20	.03	----	Atlas-Centaur - upper stage[a]
7	Lunar	Two-burn	1/1/69	114	-1.22	-----	1201	Atlas-Centaur

[a]High-energy cryogenic upper stage.

For the two-burn lunar, Venus and Mercury missions, an Atlas Centaur vehicle was simulated. The powered flight trajectory started with a boost into a 100-nautical mile parking orbit. The Atlas and Centaur were used to attain orbital altitude and velocity. The vehicle then coasted in orbit until the vehicle had "caught up" with the desired transfer conic. At this time, the Centaur stage was reignited and burned until the required final injection conditions had been achieved. Steering of the launch vehicle during Atlas booster phase was essentially zero angle of attack in order to minimize aerodynamic heating and loads. Once the vehicle has exited from the sensible atmosphere, explicit guidance equations were used to steer the vehicle into parking orbit and later onto the desired transfer trajectory. The guidance equations used are presented in reference 6.

The direct ascent lunar, Mars, and Jupiter cases used a three-stage version of the Atlas-Centaur launch vehicle. For the Jupiter case, the third stage was a Burner II, which is a small solid stage. For the lunar and Mars cases, the third stage was an assumed cryogenic upper stage.

In addition to the reference trajectory profiles, a guidance hardware error model and error values are required. The error model and error values used in all cases are presented in table I. These errors are used to calculate the errors in sensed acceleration, which in turn, are used as driving functions in the integration of the linearized equations of motion.

The results obtained are summarized in table III. Seven check cases are presented, five planetary and two lunar. For the planetary cases, the best agreement is obtained for the Mercury case, where the error is 0.2 percent. The errors for the Venus and Jupiter cases are 4 and 6 percent, respectively; the two Mars cases are in error by 3.5 and 14 percent. The analytical and N Body results agree within 2.5 percent for the lunar cases.

The Mars N Body results were obtained for midcourse correction times of 2, 3, 5, and 10 days after injection, in order to demonstrate that the required correction velocity does not change appreciably for small correction times compared to the total trip time. These results are shown in figure 3.

TABLE III. - SUMMARY OF ERROR ANALYSIS RESULTS

Case	Mission	Midcourse correction time	Figure of merit, m/sec			
			Miss plus time		Miss only	
			Analytical	N Body	Analytical	N Body
1	Mars	2.7 Days	3.90	3.84	----	----
2	Mars	2.7 Days	3.39	2.97	----	----
3	Jupiter	1.1 Days	8.07	7.59	----	----
4	Venus	2.6 Days	3.78	3.63	----	----
5	Mercury	1.6 Days	4.81	4.80	----	----
6	Lunar	20 Hr	6.11	6.15	3.57	3.66
7	Lunar	20 Hr	6.03	5.90	3.66	3.57

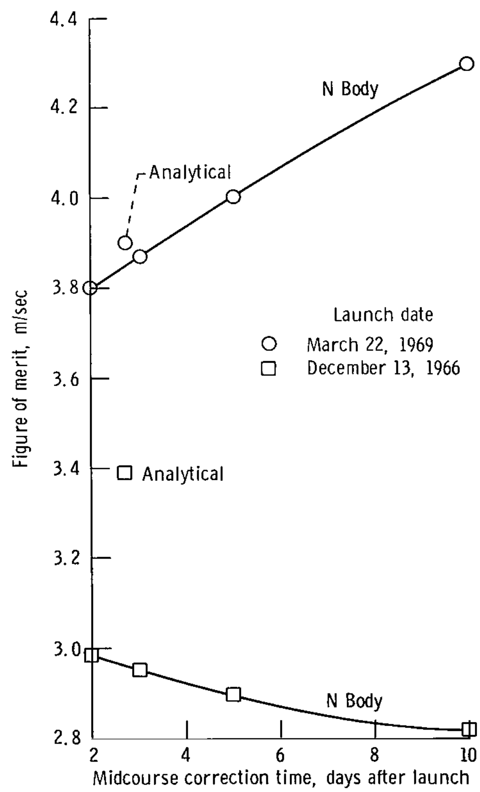

Figure 3. - Variation of required midcourse correction velocity with correction time for direct ascent Earth-Mars trajectories.

TABLE IV. - COMPARISON OF ANALYTICAL AND N BODY RESULTS

FOR VENUS TRAJECTORY - CASE 4

[Figure of merit, 3.63 m/sec for miss plus time of flight.]

(a) Analytical results

Error		Figure of merit sensitivity coefficient (miss plus time)	Figure of merit per error source (miss plus time), m/sec	Percent of figure of merit squared (miss plus time)
Number	Magnitude			
1	0.0540	2.83	0.15	0.16
2	.0540	12.06	.65	2.97
3	.0900	3.37	.30	.64
4	.0500	6.50	.32	.74
5	.0500	11.44	.57	.29
6	.0550	2.64	.15	.15
7	.0420	14.32	.60	2.53
8	.0420	4.21	.18	.22
9	.0420	11.05	.46	1.51
10	.0051	241.34	1.23	10.61
11	.0051	15.15	.08	.04
12	.0051	102.10	.52	1.90
13	.0840	5.67	.48	1.59
14	.0930	20.02	1.86	24.28
15	.0940	7.02	.66	3.05
16	.1730	4.93	.85	5.09
17	.1900	4.76	.90	5.72
18	.1770	4.14	.73	3.75
19	.1060	1.18	.12	.11
20	.1010	19.72	1.99	27.76
21	.1140	1.68	.19	.26
22	.0090	2.27	.02	.00
23	.0090	9.62	.09	.05
24	.0090	1.68	.02	.00
28	.0094	37.19	.35	.86
29	.0094	.62	.01	.00
30	.0094	21.81	.20	.29
34	.0130	6.09	.08	.04
35	.0130	.53	.01	.00
36	.0130	2.67	.03	.01
37	.0120	9.15	.11	.08
38	.0120	2.48	.03	.01
39	.0120	11.27	.14	.13
40	.0240	22.68	.54	2.07
41	.0260	6.10	.16	.18
42	.0290	11.05	.32	.72
43	.0090	2.29	.02	.00
44	.0090	10.07	.09	.06
45	.0090	14.39	.13	.12
46	.0080	1.77	.01	.00
47	.0080	1.82	.01	.00
48	.0080	3.44	.03	.01

TABLE IV. - Concluded. COMPARISON OF ANALYTICAL AND N BODY

RESULTS FOR VENUS TRAJECTORY - CASE 4

[Figure of merit, 3.78 m/sec for miss plus time of flight.]

(b) N Body results

Error		Figure of merit sensitivity coefficient (miss plus time)	Figure of merit per error source (miss plus time), m/sec	Percent of figure of merit squared (miss plus time)
Number	Magnitude			
1	0.0540	3.14	0.17	0.22
2	.0540	11.67	.63	3.02
3	.0900	1.56	.14	.15
4	.0500	6.00	.30	.68
5	.0500	10.93	.55	2.27
6	.0550	2.52	.14	.15
7	.0420	15.56	.65	3.25
8	.0420	3.77	.16	.19
9	.0420	10.58	.44	1.50
10	.0051	251.84	1.28	12.54
11	.0051	13.99	.07	.04
12	.0051	105.54	.54	2.20
13	.0840	6.61	.56	2.34
14	.0930	19.43	1.81	24.81
15	.0940	.33	.03	.01
16	.1730	5.52	.95	6.93
17	.1900	4.55	.86	5.68
18	.1770	2.09	.37	1.04
19	.1060	1.32	.14	.15
20	.1010	18.87	1.91	27.61
21	.1140	.33	.04	.01
22	.0090	2.53	.02	.00
23	.0090	9.18	.08	.05
24	.0090	.73	.01	.00
28	.0094	38.91	.37	1.02
29	.0094	.58	.01	.00
30	.0094	21.24	.20	.30
34	.0130	6.56	.09	.06
35	.0130	.73	.01	.00
36	.0130	2.82	.04	.01
37	.0120	9.58	.12	.10
38	.0120	2.33	.03	.01
39	.0120	11.81	.14	.15
40	.0240	23.65	.57	2.45
41	.0260	5.62	.15	.16
42	.0290	10.58	.31	.72
43	.0090	2.40	.02	.00
44	.0090	9.41	.08	.05
45	.0090	13.65	.12	.11
46	.0080	1.90	.02	.00
47	.0080	2.75	.02	.00
48	.0080	3.27	.03	.01

TABLE V. - COMPARISON OF ANALYTICAL AND N BODY RESULTS FOR LUNAR TRAJECTORY - CASE 6

[Figure of merit, 3.57 m/sec for miss only and 6.11 m/sec for miss plus time of flight.]

(a) Analytical results

Error		Figure of merit sensitivity coefficient (miss only)	Figure of merit per error source (miss only), m/sec	Percent of figure of merit squared (miss only)	Figure of merit sensitivity coefficient (miss plus time)	Figure of merit per error source (miss plus time), m/sec	Percent of figure of merit squared (miss plus time)
Number	Magnitude						
1	0.0540	3.88	0.21	0.34	6.24	0.34	0.30
2	.0540	15.38	.83	5.41	28.58	1.54	6.38
3	.0900	1.05	.09	.07	1.11	.10	.03
4	.0500	7.87	.39	1.21	14.92	.75	1.49
5	.0500	12.97	.65	3.30	22.30	1.12	3.33
6	.0550	2.81	.15	.19	4.84	.27	.19
7	.0420	13.11	.55	2.38	30.37	1.28	4.36
8	.0420	3.11	.13	.13	6.70	.28	.21
9	.0420	10.43	.44	1.51	17.68	.74	1.48
10	.0051	318.79	1.63	20.73	680.44	3.47	32.28
11	.0051	17.03	.09	.06	32.30	.16	.07
12	.0051	43.59	.22	.39	95.50	.49	.64
13	.0840	3.94	.33	.86	5.36	.45	.54
14	.0930	17.17	1.60	19.99	24.76	2.30	14.21
15	.0940	1.26	.12	.11	1.29	.12	.04
16	.1730	4.22	.73	4.19	5.65	.98	2.56
17	.1900	4.02	.76	4.58	5.71	1.09	3.16
18	.1770	1.33	.24	.44	1.38	.24	.16
19	.1060	.92	.10	.07	1.24	.13	.05
20	.1010	18.47	1.87	27.29	26.17	2.64	18.72
21	.1140	.30	.03	.01	.31	.04	.00
22	.0090	1.93	.02	.00	2.77	.02	.00
23	.0090	8.30	.07	.04	12.74	.11	.04
24	.0090	.42	.00	.00	.43	.00	.00
28	.0094	50.64	.48	1.78	108.47	1.02	2.79
29	.0094	.62	.01	.00	1.15	.01	.00
30	.0094	7.87	.07	.04	12.38	.12	.04
34	.0130	3.94	.05	.02	9.40	.12	.04
35	.0130	.41	.01	.00	.51	.01	.00
36	.0130	1.45	.02	.00	3.16	.04	.00
37	.0120	11.11	.13	.14	23.80	.29	.22
38	.0120	2.80	.03	.01	5.22	.06	.01
39	.0120	6.56	.08	.05	14.30	.17	.08
40	.0240	27.26	.65	3.36	58.26	1.40	5.24
41	.0260	6.80	.18	.24	12.78	.33	.30
42	.0290	10.43	.30	.72	17.68	.51	.70
43	.0090	2.45	.02	.00	5.24	.05	.01
44	.0090	12.77	.11	.10	23.81	.21	.12
45	.0090	18.52	.17	.22	30.57	.28	.20
46	.0080	.87	.01	.00	2.07	.02	.00
47	.0080	1.87	.01	.00	2.30	.02	.00
48	.0080	4.05	.03	.01	6.67	.05	.01

TABLE V. - Concluded. COMPARISON OF ANALYTICAL AND N BODY RESULTS FOR LUNAR TRAJECTORY - CASE 6

[Figure of merit, 3.66 m/sec for miss only and 6.15 m/sec for miss plus time of flight.]

(b) N Body results

Error		Figure of merit sensitivity coefficient (miss only)	Figure of merit per error source (miss only), m/sec	Percent of figure of merit squared (miss only)	Figure of merit sensitivity coefficient (miss plus time)	Figure of merit per error source (miss plus time), m/sec	Percent of figure of merit squared (miss plus time)
Number	Magnitude						
1	0.0540	3.97	0.21	0.34	6.24	0.34	0.30
2	.0540	15.91	.86	5.51	28.80	1.56	6.39
3	.0900	1.02	.09	.06	1.11	.10	.03
4	.0500	8.21	.41	1.26	15.19	.76	1.53
5	.0500	13.34	.67	3.32	22.38	1.12	3.31
6	.0550	2.89	.16	.19	4.85	.27	.19
7	.0420	13.94	.59	2.56	31.46	1.32	4.61
8	.0420	3.28	.14	.14	6.92	.29	.22
9	.0420	10.71	.45	1.51	17.70	.74	1.46
10	.0051	336.04	1.71	21.90	696.05	3.55	33.30
11	.0051	17.76	.09	.06	32.90	.17	.07
12	.0051	46.46	.24	.42	98.77	.50	.67
13	.0840	3.96	.33	.83	5.27	.44	.52
14	.0930	17.35	1.61	19.41	24.46	2.27	13.67
15	.0940	1.23	.12	.10	1.28	.12	.04
16	.1730	4.24	.73	4.01	5.55	.96	2.44
17	.1900	4.06	.77	4.43	5.64	1.07	3.03
18	.1770	1.30	.23	.40	1.37	.24	.16
19	.1060	.93	.10	.07	1.21	.13	.04
20	.1010	18.64	1.88	26.42	25.81	2.61	17.96
21	.1140	.30	.03	.01	.31	.03	.00
22	.0090	1.95	.02	.00	2.73	.02	.00
23	.0090	8.43	.08	.04	12.65	.11	.03
24	.0090	.41	.00	.00	.43	.00	.00
28	.0094	53.38	.50	1.88	110.96	1.04	2.87
29	.0094	.64	.01	.00	1.17	.01	.00
30	.0094	8.01	.08	.04	12.30	.12	.04
34	.0130	4.15	.05	.02	9.60	.12	.04
35	.0130	.41	.01	.00	.50	.01	.00
36	.0130	1.54	.02	.00	3.26	.04	.00
37	.0120	11.71	.14	.15	24.34	.29	.23
38	.0120	2.92	.04	.01	5.31	.06	.01
39	.0120	6.99	.08	.05	14.78	.18	.08
40	.0240	28.73	.69	3.55	59.60	1.43	5.41
41	.0260	7.08	.18	.25	13.01	.34	.30
42	.0290	10.71	.31	.72	17.70	.51	.70
43	.0090	2.58	.02	.00	5.36	.05	.01
44	.0090	13.29	.12	.11	24.22	.22	.13
45	.0090	18.95	.17	.22	30.53	.27	.20
46	.0080	.91	.01	.00	2.11	.02	.00
47	.0080	1.87	.01	.00	2.28	.02	.00
48	.0080	4.14	.03	.01	6.66	.05	.01

TABLE VI. - FIGURE OF MERIT COVARIANCE MATRIX (MISS PLUS TIME) FOR VENUS TRAJECTORY - CASE 4

(a) Analytical partials

0.7345719×10^{-5}	0.5974771×10^{-6}	-0.1421085×10^{-12}	0.1587145×10^{-2}	0.8998715×10^{-2}	-0.7227641×10^{-10}
$.5974771\times10^{-6}$	$.1997425\times10^{-6}$	$-.2664535\times10^{-14}$	$.3208562\times10^{-3}$	$.6506040\times10^{-3}$	$-.1032960\times10^{-9}$
$-.1421085\times10^{-12}$	$-.2664535\times10^{-14}$	$.1997425\times10^{-6}$	$-.1046427\times10^{-7}$	$-.1594759\times10^{-6}$	$.6915269\times10^{-4}$
$.1587145\times10^{-2}$	$.3208562\times10^{-3}$	$-.1046427\times10^{-7}$	.5862252	1.8413685	$-.1117587\times10^{-7}$
$.8998715\times10^{-2}$	$.6506040\times10^{-3}$	$-.1594759\times10^{-6}$	1.8413685	11.067776	$-.2384186\times10^{-6}$
$-.7227641\times10^{-10}$	$-.1032960\times10^{-9}$	$.6915269\times10^{-4}$	$-.1117587\times10^{-7}$	$-.2384186\times10^{-6}$	$.2394136\times10^{-1}$

(b) N Body partials

0.7456713×10^{-5}	0.6651150×10^{-6}	-0.6675128×10^{-7}	0.1764756×10^{-2}	0.9074949×10^{-2}	0.3023507×10^{-4}
$.6651150\times10^{-6}$	$.2207393\times10^{-6}$	$-.1500603\times10^{-7}$	$.3617413\times10^{-3}$	$.7209460\times10^{-3}$	$.8065659\times10^{-5}$
$-.6675128\times10^{-7}$	$-.1500603\times10^{-7}$	$.2069464\times10^{-6}$	$-.2712879\times10^{-4}$	$-.7663904\times10^{-4}$	$-.1109500\times10^{-3}$
$.1764756\times10^{-2}$	$.3617413\times10^{-3}$	$-.2712879\times10^{-4}$	.6763185	2.0356938	$.1499715\times10^{-1}$
$.9074949\times10^{-2}$	$.7209460\times10^{-3}$	$-.7663904\times10^{-4}$	2.0356938	11.092905	$.4544180\times10^{-1}$
$.3923507\times10^{-4}$	$.8065659\times10^{-5}$	$-.1109500\times10^{-3}$	$.1499715\times10^{-1}$	$.4544180\times10^{-1}$	$.5948564\times10^{-1}$

TABLE VII. - FIGURE OF MERIT COVARIANCE MATRIX FOR LUNAR TRAJECTORY - CASE 6

(a) Analytical partials; miss plus time

0.4674652×10^{-4}	0.1270308×10^{-5}	-0.6821210×10^{-12}	0.1515782×10^{-2}	0.5531167×10^{-1}	-0.1564331×10^{-9}
$.1270308\times10^{-5}$	$.1648680\times10^{-6}$	$-.7105427\times10^{-14}$	$.1958213\times10^{-3}$	$.1488603\times10^{-2}$	$-.4684253\times10^{-11}$
$-.6821210\times10^{-12}$	$-.7105427\times10^{-14}$	$.1543766\times10^{-6}$	$-.3613799\times10^{-8}$	$-.5540254\times10^{-6}$	$-.1602241\times10^{-4}$
$.1515782\times10^{-2}$	$.1958213\times10^{-3}$	$-.3613799\times10^{-8}$	$.2325870$	1.7761296	$-.4656613\times10^{-8}$
$.5531167\times10^{-1}$	$.1488603\times10^{-2}$	$-.5540254\times10^{-6}$	1.7761296	65.448912	$-.9536743\times10^{-6}$
$-.1564331\times10^{-9}$	$-.4684253\times10^{-11}$	$-.1602241\times10^{-4}$	$-.4656613\times10^{-8}$	$-.9536743\times10^{-6}$	$.1662929\times10^{-2}$

(b) Analytical partials; miss only

0.1131277×10^{-4}	-0.2109825×10^{-6}	0.1002750×10^{-5}	-0.2481504×10^{-3}	0.1344401×10^{-1}	-0.1040733×10^{-3}
$.2109825\times10^{-6}$	$.1029433\times10^{-6}$	$.4191951\times10^{-7}$	$.1220812\times10^{-3}$	$-.2620735\times10^{-3}$	$-.4350733\times10^{-5}$
$.1002750\times10^{-5}$	$.4191951\times10^{-7}$	$.1259994\times10^{-6}$	$.4991549\times10^{-4}$	$.1184639\times10^{-2}$	$-.1307721\times10^{-4}$
$-.2481504\times10^{-3}$	$.1220812\times10^{-3}$	$.4991549\times10^{-4}$	$.1447765$	$-.3081134$	$-.5180882\times10^{-2}$
$.1344401\times10^{-1}$	$-.2620735\times10^{-3}$	$.1184639\times10^{-2}$	$-.3081134$	15.977962	$-.1229724$
$-.1040733\times10^{-3}$	$-.4350733\times10^{-5}$	$-.1307721\times10^{-4}$	$-.5180882\times10^{-2}$	$-.1229724$	$.1357254\times10^{-2}$

(c) N Body partials; miss plus time

0.4869086×10^{-4}	0.1181656×10^{-5}	-0.1495929×10^{-6}	0.1791531×10^{-2}	0.5762047×10^{-1}	0.1037888×10^{-4}
$.1181656\times10^{-5}$	$.1584029\times10^{-6}$	$-.5109898\times10^{-8}$	$.1973609\times10^{-3}$	$.1382835\times10^{-2}$	$.2036971\times10^{-6}$
$-.1495929\times10^{-6}$	$-.5109898\times10^{-8}$	$.1550908\times10^{-6}$	$-.7269783\times10^{-5}$	$-.1770799\times10^{-3}$	$-.1587410\times10^{-4}$
$.1791531\times10^{-2}$	$.1973609\times10^{-3}$	$-.7269783\times10^{-5}$	$.2484564$	2.1016650	$.3258169\times10^{-3}$
$.5762047\times10^{-1}$	$.1382835\times10^{-2}$	$-.1770799\times10^{-3}$	2.1016650	68.189584	$.1231161\times10^{-1}$
$.1037888\times10^{-4}$	$.2036971\times10^{-6}$	$-.1587410\times10^{-4}$	$.3258169\times10^{-3}$	$.1231161\times10^{-1}$	$.1625580\times10^{-2}$

(d) N Body partials; miss only

0.1247782×10^{-4}	-0.2527232×10^{-6}	0.1000962×10^{-5}	-0.1999562×10^{-3}	0.1483115×10^{-1}	-0.1029777×10^{-3}
$-.2527232\times10^{-6}$	$.1015879\times10^{-6}$	$.4046299\times10^{-7}$	$.1184792\times10^{-3}$	$-.3120274\times10^{-3}$	$-.4286299\times10^{-5}$
$.1000962\times10^{-5}$	$.4046299\times10^{-7}$	$.1185355\times10^{-6}$	$.5600343\times10^{-4}$	$.1182416\times10^{-2}$	$-.1227255\times10^{-4}$
$-.1999562\times10^{-3}$	$.1184792\times10^{-3}$	$.5600343\times10^{-4}$	$.1389373$	$-.2514756$	$-.5908076\times10^{-2}$
$.1483115\times10^{-1}$	$-.3120274\times10^{-3}$	$.1182416\times10^{-2}$	$-.2514756$	17.629724	$-.1216305$
$-.1029777\times10^{-3}$	$-.4286299\times10^{-5}$	$-.1227255\times10^{-4}$	$-.5908076\times10^{-2}$	$-.1216305$	$.1270743\times10^{-2}$

The FOM results presented in table III are obtained by statistically combining the effects of the guidance error sources presented in table I. The contribution of each error source to the total FOM is presented for Venus and lunar missions (cases 4 and 6) in tables IV and V. In addition to the FOM per error source, the FOM sensitivity coefficient (FOM per unit error source) and percent of FOM squared are also presented for each error source. These quantities allow the FOM to be quickly recalculated for different error values. The analytical and N Body FOM covariance matrices are also presented for the same two cases in tables VI and VII.

Sample Problem

Suppose it is desired to determine the trajectory resulting in the largest FOM for a particular planetary launch opportunity. Such a problem arises in preliminary design studies, where it is necessary to size the midcourse propulsion system. If the analytical equations described in this report are used, the possible trajectories for the opportunity may be described in terms of the powered flight trajectory variables. For a two-burn mission, these variables are launch azimuth, energy, and parking orbit coast time. For a direct ascent mission, injection true anomaly replaces coast time as a variable. The largest FOM can then be determined by searching all possible values of the powered flight variables, the range of which may be found in the literature; for example, references 4 and 5.

The use of this technique is illustrated on figure 4 for a two-burn Mars mission. Figure of merit is plotted against injection energy for different values of parking orbit coast time. The launch azimuth is constant on figure 4; therefore, other figures must be generated for different launch azimuths. These figures are not presented here. The largest FOM may be easily determined from the figures.

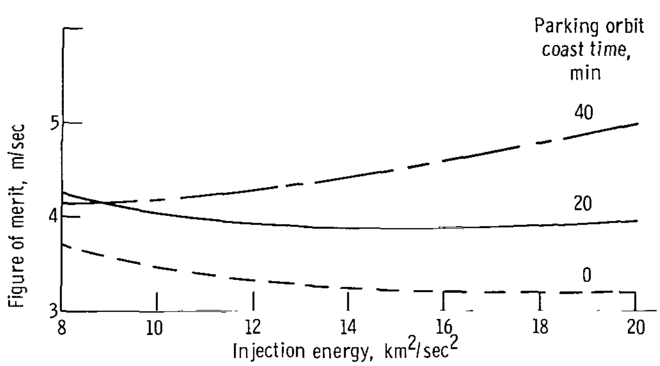

Figure 4. - Variation of figure of merit with energy and coast time. Launch azimuth, constant; two-burn Mars mission.

After the values of the powered flight variables resulting in the largest FOM have been determined, an N Body trajectory corresponding to this case can be generated, if a more accurate estimate of FOM is desired. Even if the analytical FOM is in error by a few percent, the trends resulting in the choice of the worst case powered flight variables can be assumed to be correct.

CONCLUSIONS

Analytical equations have been developed for computing partial derivatives relating midcourse correction velocity requirements to injection errors for planetary and lunar missions. These partial derivatives are used to calculate the guidance system FOM. The equations are easily programmed on a digital computer and require very little computer time. In addition, no free flight reference trajectories are required. Thus, error analysis results can be obtained without first having to calculate detailed targeted trajectories, a process which requires a powered flight-free flight (N Body) targeting program.

Analytical and N Body results have been generated for two lunar and five planetary missions. These results show good agreement between the analytical and N Body results. For the planetary cases, the best agreement is obtained for the Mercury case where the error is 0.2 percent. The errors for the other planetary cases vary between 2 and 14 percent. For the lunar cases, the results agree within 2.5 percent.

Because of the accuracy of the results presented and the ease with which these results may be obtained, the analytical equations provide a useful preliminary design tool for estimating space mission FOM. The analytical technique can also be readily extended to apply to other mission types.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, January 12, 1968,
125-17-05-01-22.

APPENDIX A

SYMBOLS

A, B, C, D	Danby's matrices
$\bar{a}$	thrust acceleration vector, m/sec^2
c	$\cos \eta$ (appendix B)
E	energy, m^2/sec^2
$E[\ ]$	expected value operator
e	eccentricity
e_k	k^{th} guidance hardware error source
G_{ik}	matrix of partial derivatives, $\partial v_i/\partial S_k$
h	angular momentum, m^2/sec
L, M, P, Q	Danby's matrices
P_{ik}	matrix of partial derivatives, $\partial S_i/\partial e_k$
p	semilatus rectum, m
r	radius, m
$\hat{r}, \hat{\theta}, \hat{h}$	unit radial, tangential, normal coordinate system
S	state variable
s	$\sin \eta$ (appendix B)
t	time
$\overline{U}, \overline{V}$	vectors defined in equations (8) and (10)
$\bar{v}$	velocity, m/sec
Δv_c	magnitude of midcourse correction velocity, m/sec
$\Delta \bar{v}_c$	midcourse correction velocity, m/sec
$\hat{z}$	unit vector pointing at North Pole
α, β, ρ	coefficients used in appendix C
γ	flight path angle, rad
$\Delta(\ )$	error quantity
$\delta(\ )$	linearized error quantity
η	true anomaly, rad
Λ	FOM covariance matrix
μ	central body gravitational constant, m^3/sec^2
ν	variable defined in appendix B, $(sec)^{-1}$
σ	injection covariance matrix
φ	latitude of moon at arrival with respect to earth's equator, rad
ψ	inclination of lunar orbit to earth's equator, rad

Subscripts:

A	arrival
a	asymptote
act	actual
c	midcourse correction
e	earth escape

h normal

I injection

M miss only

MT miss plus time

m moon

mes measured by the guidance system

nom nominal

p perigee

r radial

t tangential

Superscripts:

T matrix transpose

$\bar{\ }$ vector

$\hat{\ }$ unit vector

General:

$\cdot$ vector dot product

$\times$ vector cross product

APPENDIX B

DANBY'S METHOD AND MATRICES

Errors in position and velocity at any time t on a coast ellipse are related to errors at periapse by the following matrix:

$$\begin{pmatrix} \delta\bar{r} \\ \\ \delta\bar{v} \end{pmatrix}_t = \begin{pmatrix} L(t, t_p) & M(t, t_p) \\ \\ P(t, t_p) & Q(t, t_p) \end{pmatrix} \begin{pmatrix} \delta\bar{r} \\ \\ \delta\bar{v} \end{pmatrix}_{t_p} \tag{B1}$$

where L, M, P, and Q are 3×3 matrices, the elements of which are presented later in this appendix as functions of e, r_p, r, η, μ, and $(t - t_p)$. To relate times t_1 and t_2 (neither at periapse), the following relation is applied:

$$\begin{pmatrix} \delta\bar{r} \\ \\ \delta\bar{v} \end{pmatrix}_{t_2} = \begin{pmatrix} A(t_2, t_1) & B(t_2, t_1) \\ \\ C(t_2, t_1) & D(t_2, t_1) \end{pmatrix} \begin{pmatrix} \delta\bar{r} \\ \\ \delta\bar{v} \end{pmatrix}_{t_1} \tag{B2}$$

where A, B, C, and D are 3×3 matrices and are of the form:

$$\left.\begin{aligned} A(t_2, t_1) &= L(t_2, t_p) \quad Q^T(t_1, t_p) - M(t_2, t_p) \quad P^T(t_1, t_p) \\ B(t_2, t_1) &= M(t_2, t_p) \quad L^T(t_1, t_p) - L(t_2, t_p) \quad M^T(t_1, t_p) \\ C(t_2, t_1) &= P(t_2, t_p) \quad Q^T(t_1, t_p) - Q(t_2, t_p) \quad P^T(t_1, t_p) \\ D(t_2, t_1) &= Q(t_2, t_p) \quad L^T(t_1, t_p) - P(t_2, t_p) \quad M^T(t_1, t_p) \end{aligned}\right\} \tag{B3}$$

The elements of the L, M, P, and Q matrices are given by:

$$L_{11} = \frac{r}{r_p(1 - e)} \left[c^2 + c(2 - e) - 2\right] + \frac{3\nu}{(1 - e)\,(1 + e)^{1/2}}\, s(t - t_p)$$

$$L_{12} = \frac{r}{r_p(1 + e)} \; s(1 - c)$$

$$L_{21} = \frac{r}{r_p(1 - e)} \; s(c + 2) - \frac{3\nu}{(1 - e)\,(1 + e)^{1/2}} \; (c + e)\,(t - t_p)$$

$$L_{22} = \frac{r}{r_p(1 + e)} \left[c^2 - c(1 - e) + 1\right]$$

$$L_{33} = \frac{r}{r_p} \; c$$

$$M_{11} = \frac{r}{\nu r_p(1 + e)^{3/2}} \; s(-c + 2 + e)$$

$$M_{12} = \frac{r}{\nu r_p(1 - e)\,(e + 1)^{1/2}} \; 2(c - 1)\,(c + 2) + \frac{3}{1 - e} \; s(t - t_p)$$

$$M_{21} = \frac{r}{\nu r_p(1 + e)^{3/2}} \; (1 - c)^2$$

$$M_{22} = \frac{r}{\nu r_p(1 - e)\,(1 + e)^{1/2}} \; 2s(c + 1 + e) - \frac{3}{1 - e} \; (c + e)\,(t - t_p)$$

$$M_{33} = \frac{r}{\nu r_p(e + 1)^{1/2}} \; s$$

$$P_{11} = \frac{\nu}{(1 - e)\,(e + 1)^{1/2}} \; s(-ec^2 - 2c + 1 - e) + \frac{3\nu^2 r_p^2}{r^2(1 - e)} \; c(t - t_p)$$

$$P_{12} = \frac{\nu}{(1 + e)^{3/2}} \; (-ec^3 - 2c^2 + c + 1 + e)$$

$$P_{21} = \frac{\nu}{(1 - e)(e + 1)^{1/2}} (ec^3 + 2c^2 - c - 1 - e) + \frac{3\nu^2 r_p^2}{r^2(1 - e)} (t - t_p)$$

$$P_{22} = \frac{\nu}{(1 + e)^{3/2}} s(-ec^2 - 2c + 1)$$

$$P_{33} = \frac{-\nu}{(1 + e)^{1/2}} s$$

$$Q_{11} = \frac{1}{(1 + e)^2} \left[-ec^3 - 2c^2 + 2c + ec + (1 + e)^2\right]$$

$$Q_{12} = \frac{1}{(1 - e)^2} s(-2ec^2 - 4c + 1 - e) + \frac{3\nu r_p^2}{r^2(1 - e)} (1 + e)^{1/2} c(t - t_p)$$

$$Q_{21} = \frac{1}{(1 + e)^2} s(1 - c)(ec + 2 + e)$$

$$Q_{22} = \frac{1}{(1 - e^2)} \left[2ec^3 + 4c^2 - c(1 + e) - 2 - e - e^2\right] + \frac{3\nu r_p^2}{r^2(1 - e)} (1 + e)^{1/2} s(t - t_p)$$

$$Q_{33} = \frac{1}{1 + e} (c + e)$$

where

$$s = \sin \eta$$

$$c = \cos \eta$$

$$\nu = \sqrt{\frac{\mu}{r_p^3}}$$

All matrix elements not listed are equal to zero.

APPENDIX C

ANALYTICAL EQUATIONS FOR CALCULATION OF FOM COVARIANCE MATRIX

Planetary case:

Calculate reference trajectory variables

$$\left.\begin{aligned}
\bar{h} &= \bar{r} \times \bar{v} \\
p &= \frac{h^2}{\mu} \\
E &= \frac{v^2}{2} - \frac{\mu}{r} \\
e &= \sqrt{1 + \frac{2Ep}{\mu}} \\
\eta_a &= \cos^{-1}\left(-\frac{1}{e}\right) \\
\eta_I &= \sin^{-1}\left(\sqrt{\frac{p}{\mu}}\,\frac{\bar{r}\cdot\bar{v}}{er}\right) \\
r_p &= \frac{p}{1+e} \\
t_I - t_p &= \sqrt{\frac{p^3}{\mu}}\left(\frac{1}{1-e^2}\right)\sqrt{\frac{1}{e^2-1}}\left[\log\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan\frac{\eta_I}{2}}{\sqrt{e+1}-\sqrt{e-1}\tan\frac{\eta_I}{2}}\right) - \frac{e\sin\eta_I}{1+e\cos\eta_I}\right] \\
t_a - t_p &= 0
\end{aligned}\right\} \quad \text{(C1)}$$

Calculate Danby's matrices L_{Ip}, M_{Ip}, P_{Ip}, Q_{Ip}, L_{ap}, M_{ap}, P_{ap}, Q_{ap} using the last equation in appendix B.

Calculate radial coordinate system at perigee.

$$\left.\begin{aligned} \hat{r}_p &= \frac{\bar{v} \times \bar{h}}{e\mu} - \frac{\bar{r}}{er} \\ \hat{h}_p &= \frac{\bar{r} \times \bar{v}}{|\bar{r} \times \bar{v}|} \\ \hat{\theta}_p &= \hat{h}_p \times \hat{r}_p \end{aligned}\right\} \tag{C2}$$

Calculate Danby's matrices C_{aI} and D_{aI} using equation (B3) in appendix B.

$$G = (C_{aI} | D_{aI})$$

$$\Lambda = G^T G$$

The injection error vector must be expressed in the perigee coordinate system

$$\delta\bar{S} = \begin{pmatrix} \delta\bar{r} \cdot \hat{r}_p \\ \delta\bar{r} \cdot \hat{\theta}_p \\ \delta\bar{r} \cdot \hat{h}_p \\ \delta\bar{v} \cdot \hat{r}_p \\ \delta\bar{v} \cdot \hat{\theta}_p \\ \delta\bar{v} \cdot \hat{h}_p \end{pmatrix} \tag{C3}$$

Lunar case:

Calculate reference trajectory variables using equations (C1) except that the equations for η_a, $(t_I - t_p)$, and $(t_a - t_p)$ are replaced by

$$\eta_A = \cos^{-1}\left(\frac{\frac{p}{r_m} - 1}{e}\right) \tag{C4}$$

$$t_I - t_p = \sqrt{\frac{p^3}{\mu}}\left(\frac{1}{1 - e^2}\right)\frac{2}{\sqrt{1 - e^2}}\left[\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\frac{\eta_I}{2}\right) - \frac{e \sin \eta_I}{1 + e \cos \eta_I}\right] \tag{C5}$$

$t_A - t_p$ same as equation (C5) but with η_A instead of η_I

$t_c - t_p$ same as equation (C5) but with η_c instead of η_I

Iterate on η_c to satisfy $t_c - t_I$ equals the desired midcourse correction time.

Calculate Danby's matrices L_{Ip}, M_{Ip}, P_{Ip}, Q_{Ip}, L_{cp}, M_{cp}, P_{cp}, Q_{cp}, L_{mp}, M_{mp}, P_{mp}, and Q_{mp}.

Calculate radial coordinate system at perigee using equations (C2). Calculate position and velocity at lunar arrival.

$$\bar{r}_A = \left(\frac{p}{1 + e \cos \eta_A}\right)(\cos \eta_A \hat{r}_p + \sin \eta_A \hat{\theta}_p)$$

$$\bar{v}_A = \frac{e}{p}\,\bar{h} \times \hat{r}_p + \frac{\bar{h} \times \bar{r}_A}{p r_A}$$

Calculate lunar velocity vector

$$\bar{v}_m = \sqrt{\frac{\mu}{r_A}}\left[\frac{-\cos \psi}{\cos \varphi}\,\hat{r}_m \times \hat{z} \pm \sqrt{\frac{1 - \cos^2\psi}{\cos^2\varphi}}\,\hat{r}_m \times (\hat{r}_m \times \hat{z})\right]$$

where

$$\sin \varphi = \hat{r}_m \cdot \hat{z}$$

$$\cos \varphi = \sqrt{1 - \sin^2\varphi}$$

Express $(\bar{v}_A - \bar{v}_m)$ in radial-perigee coordinate system.

$$(\bar{v}_A - \bar{v}_m) = \begin{matrix} (\bar{v}_A - \bar{v}_m) \cdot \hat{r}_p \\ (\bar{v}_A - \bar{v}_m) \cdot \hat{\theta}_p \\ (\bar{v}_A - \bar{v}_m) \cdot \hat{h}_p \end{matrix}$$

Calculate Danby's matrices A_{mI}, B_{mI}, and B_{mc} using equations (B3) in appendix B.

Miss plus time correction:

$$G = -\left(B_{mc}^{-1}A_{mI} \middle| B_{mc}^{-1}B_{mI}\right)$$

$$\Lambda_{MT} = G^T G$$

Miss only correction:

$$\overline{V} = -B_{mc}^{-1}(\overline{v}_A - \overline{v}_m)$$

$$\Lambda_M = \Lambda_{MT} - \frac{G^T \overline{V}\,\overline{V}^T G}{\overline{V}^T \overline{V}}$$

The injection error vector is expressed in the perigee-radial coordinate system, as in equation (C3).

APPENDIX D

CALCULATION OF UNIT LUNAR VELOCITY VECTOR, $\hat{v}_m$

Let $\hat{v}_m$ be expressed as

$$\hat{v}_m = \alpha\hat{r}_m + \beta(\hat{r}_m \times \hat{z}) + \rho\hat{r}_m \times (\hat{r}_m \times \hat{z}) \tag{D1}$$

Three equations are available for determining α, β, and ρ based on the assumption that the moon's orbit about the earth is circular with an inclination ψ to the earth's equator.

$$\hat{v}_m \cdot \hat{v}_m = 1 \tag{D2}$$

$$\hat{v}_m \cdot \hat{r}_m = 0 \tag{D3}$$

$$(\hat{r}_m \times \hat{v}_m) \cdot \hat{z} = -\hat{v}_m \cdot (\hat{r}_m \times \hat{z}) = \cos\psi \tag{D4}$$

Combining equations (D1) and (D2) results in

$$\alpha^2 + \beta^2(\hat{r}_m \times \hat{z})^2 + \rho^2\left[\hat{r}_m \times (\hat{r}_m \times \hat{z})\right]^2 = 1 \tag{D5}$$

since the three vectors used in defining $\hat{v}_m$ are mutually orthogonal. But

$$(\hat{r}_m \times \hat{z})^2 = 1 - (\hat{r}_m \cdot \hat{z})^2 = 1 - \sin^2\varphi = \cos^2\varphi \tag{D6}$$

where φ is the latitude of the moon relative to the earth. Also,

$$\left[\hat{r}_m \times (\hat{r}_m \times \hat{z})\right]^2 = (\hat{r}_m \times \hat{z})^2 = \cos^2\varphi$$

Therefore, equation (D5) becomes

$$\alpha^2 + (\beta^2 + \rho^2)\cos^2\varphi = 1 \tag{D7}$$

From equations (D1) and (D3) it follows that

$$\hat{r}_m \cdot \hat{v}_m = (\hat{r}_m \cdot \hat{r}_m)\alpha = 0$$

which implies that

$$\alpha = 0 \tag{D8}$$

From equations (D1) and (D4)

$$\hat{v}_m \cdot (\hat{r}_m \times \hat{z}) = \beta(\hat{r}_m \times \hat{z})^2 = -\cos \psi$$

or

$$\beta = \frac{-\cos \psi}{\cos^2 \varphi} \tag{D9}$$

where equation (D6) has been used. Combining equations (D7) and (D9) results in

$$\rho = \frac{\pm 1}{\cos \varphi} \sqrt{1 - \frac{\cos^2 \psi}{\cos^2 \varphi}} \tag{D10}$$

Substituting equations (D8) to (D10) into equation (D1) results in

$$\hat{v}_m = \frac{1}{\cos \varphi} \left[- \frac{\cos \psi}{\cos \varphi} (\hat{r}_m \times \hat{z}) \pm \sqrt{1 - \frac{\cos^2 \psi}{\cos^2 \varphi}} \; \hat{r}_m \times (\hat{r}_m \times \hat{z}) \right] \tag{D11}$$

The ambiguity in sign in equation (D11) indicates that there exist two possible velocity vectors satisfying equations (D2) to (D4). These two possibilities correspond to a rising or setting moon. Taking the dot product of equation (D11) with $\hat{z}$ results in

$$\hat{v}_m \cdot \hat{z} = \pm \frac{1}{\cos \varphi} \sqrt{1 - \frac{\cos^2 \psi}{\cos^2 \varphi}} \left[(\hat{r}_m \cdot \hat{z})\hat{r}_m - \hat{z} \right] \cdot \hat{z} = \mp \sqrt{\cos^2 \varphi - \cos^2 \psi} \tag{D12}$$

Therefore, the sign choices in equation (D11) correspond to

+ sign   indicates descending moon

- sign   indicates ascending moon

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